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On differential structure for projective limits of manifolds

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Abstract

We investigate the differential calculus defined by Ashtekar and Lewandowski on projective limits of manifolds by means of cylindrical smooth functions and compare it with the $\mathcal{C}^\infty$ calculus proposed by Fröhlicher and Kriegl in a more general context. For products of connected manifolds, a Boman theorem is proved, showing the equivalence of the two calculi in this particular case. Several examples of projective limits of manifolds are discussed, arising in String Theory and in loop quantization of Gauge Theories. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the recent literature in mathematical physics one often encounters spaces which are projective limits of manifolds. In the loop quantization of Gauge Theories as Quantum Gravity and the 2D Yang–Mills Theory, projective families of manifolds are widely used to obtain a compact space $\overline{\mathcal{A}/\mathcal{G}}$ extending the space of connections modulo gauge transformations. This procedure allows one to define a diffeomorphism invariant measure on $\overline{\mathcal{A}/\mathcal{G}}$ in order to get a Hilbert representation of Wilson loop observables ([3,4,8]; for a general reference for Loop Quantum Gravity, see also the bibliography in [43]).

Another example arises in String Theory. Actually, Nag and Sullivan considered in [41] the projective family of all finite sheeted compact unbranched coverings of a given closed

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Riemann surface of genus $g \geq 2$, obtaining a universal object, called the universal hyperbolic solenoid. To this projective limit of surfaces corresponds the universal Teichmüller space $\mathcal{T}_\infty$, the inductive limit of the family of Teichmüller spaces on each surface. $\mathcal{T}_\infty$ contains the Teichmüller spaces of surfaces of every genus $g \geq 2$, so that it could simultaneously parametrize complex structures on surfaces of all topologies and it has been proposed as a fundamental object for a nonperturbative quantization of String Theory [40].

One can ask whether projective limits of manifolds admit a suitable differentiable structure. Among projective limits of manifolds there are manifolds (ordinary or modelled on infinite-dimensional spaces) and spaces which are not manifolds. Examples of such pathology are compact groups. The notion of projective limit was introduced by Weil [48] just to discuss the structure of locally compact groups and Weil himself proved that every compact group is the projective limit of a family of compact Lie groups. This does not longer mean that any compact group admits some differential structure. Actually, a projective limit of a nontrivial family of compact Lie groups cannot be a Lie group. What is worse, it is well known that compact groups can have a wild topological structure. This example shows that, if one does research for a differential structure on projective limits of manifolds, one is forced to a profound enlargement of the usual notions of differential structure, still remaining on commutative differential calculi.

This problem seems not so evident in the case of the hyperbolic solenoid introduced by Nag and Sullivan, since this space is just a foliated surface, a well-understood differential structure [37]. There are serious physical motivations to introduce a differential calculus and differential operators on projective limits arising in loop quantization. These limits can be very different, so that a general treatment appears to be necessary. A solution of the problem has been proposed by Ashtekar and Lewandowski in [5] by choosing as ring of smooth functions the set of the cylindrical smooth functions. Roughly speaking, on a projective limit of manifolds $M = \varprojlim_{j \in J} M_j$, one considers to be apt to differential calculus just the smooth functions on some manifold M_j of the family. Cylindrical differential forms, vector fields and other differential objects are consequently defined. In Section 2 we introduce projective limits of manifolds, give a short account of Ashtekar–Lewandowski calculus, set up tangent bundles and give some simple examples.

In the mathematical literature several attempts to generalize differential calculus and the notion of differential manifold can be found [19,26,36,38]. In this paper we compare the calculus proposed by Ashtekar and Lewandowski with the C^∞ calculus, developed by Frölicher and Kriegl in [19], of which we give a short account in Section 3. The C^∞ calculus assumes as starting point the duality between smooth curves and smooth functions expressed by the Boman theorem for ordinary manifolds [12]:

(1) for every ordinary manifold M a path $c : \mathbf{R} \to M$ is smooth if and only if $f \circ c$ is smooth for every smooth function $f : M \to \mathbf{R}$;

(2) a map $\varphi : M \to N$, where N is an ordinary manifold, is smooth if and only if $\varphi \circ c$ is a smooth curve in N for every smooth curve c in M.

A C^∞ structure on a set X is accordingly defined assigning a suitable set of "curves" $c : \mathbf{R} \to X$ and a suitable set of "functions" $f : X \to \mathbf{R}$ such that $f \circ c \in C^\infty(\mathbf{R}, \mathbf{R})$. The

$\mathcal{C}^\infty$ category contains ordinary manifolds and has many nice mathematical properties; in particular, it is Cartesian closed and closed with respect to projective limits. The $\mathcal{C}^\infty$ calculus has been proved fruitful in locally convex vector spaces where straight lines assure a richness of curves to get a good differential calculus. Besides, the notion of $\mathcal{C}^\infty$ structure and $\mathcal{C}^\infty$ maps revealed useful in Gauge Theory to characterize the holonomy maps associated to smooth connections [31], as reported in Example 7, Section 3, even if differential calculus is not developed in this setting. In the general case the $\mathcal{C}^\infty$ category appears too large to treat differential calculus, since the extension of the class of $\mathcal{C}^\infty$ functions depends on the richness of curves and the theory works, as it stands, only when a balance between curves and $\mathcal{C}^\infty$ functions is assured. Otherwise one clashes with an excess of $\mathcal{C}^\infty$ functions and with the difficulty of defining a good differential for every $\mathcal{C}^\infty$ function. In a general context some additional requirements on the duality between curves and functions could be necessary.

For projective limits of manifolds, cylindrical smooth functions are just a generating set for the canonical $\mathcal{C}^\infty$ structure and the class of $\mathcal{C}^\infty$ functions can be remarkably larger. The ring of cylindrical smooth functions appears as the minimal choice of functions to be considered in differential calculus, and the ring of $\mathcal{C}^\infty$ functions the maximal one.

There are examples of projective limits of manifolds where $\mathcal{C}^\infty$ calculus works well and defines the natural differential calculus: for instance, the spaces $\mathbf{R}^\mathbf{N}$ and the manifold $J^\infty(M, N)$ of jets of infinite order of maps between two ordinary manifolds M and N, introduced in Section 2 and discussed in Section 3. These spaces are Fréchet manifolds, on which the $\mathcal{C}^\infty$ calculus gives the standard differential calculus. Here the choice of cylindrical smooth functions appears as an unnecessary, even if not severe, restriction.

The two calculi agree in the particular case of products of compact connected manifolds. Actually, in Section 4 we prove a Boman theorem for cylindrical smooth functions on such products.

In Section 5 we give a first characterization of cylindrical smooth functions in terms of the $\mathcal{C}^\infty$ structure for projective limits of compact connected manifolds. One cannot expect that $\mathcal{C}^\infty$ functions are cylindrical in the general case. The most relevant obstruction is the occurrence of many path components which could cause a plenty of $\mathcal{C}^\infty$ functions. Some examples of projective limits of compact connected manifolds are discussed which support a natural structure of foliated manifold, as the hyperbolic solenoid. In these cases the appropriate ring of "smooth" functions lies between the ring of cylindrical smooth functions and the ring of $\mathcal{C}^\infty$ functions.

Finally, we give a short account of the projective limits introduced in Gauge Theory to obtain a compact space $\overline{\mathcal{A}}$ extending the space of smooth connections $\mathcal{A}$ [5]. This is the most interesting case of projective limits of manifolds, since the projections are highly not standard, so that the relation between $\mathcal{C}^\infty$ and cylindrical smooth functions is difficult to establish. One could use suitable projective subfamilies or different projective families to obtain a compact extension of $\mathcal{A}$, chosen on the basis of physical and mathematical criteria. We suggest that a natural mathematical requirement to select these families could be the possibility to obtain a satisfactory version of Boman theorem.

2. Projective limits of manifolds

We start with some standard facts about projective families and projective limits of (Hausdorff) topological spaces (see [17] or [18]).

A *projective family* of topological spaces is a family $\{X_j, \pi_{ij}, J\}$, where the index set J is a directed set, X_j is a topological space for each $j \in J$ and the projections $\pi_{ij} : X_j \to X_i$, defined for every pair $i, j \in J$ with $i \leq j$, are continuous maps such that $\pi_{jj} = id_{X_j}$ and $\pi_{ij}\pi_{jk} = \pi_{ik}$ for $i \leq j \leq k$.

An element $\{x_j\}_{j\in J}$ of the product $\prod_{j\in J} X_j$ is called a *thread* if $\pi_{ij}x_j = x_i$ for $i < j$. The set $X = \varprojlim_{j\in J} X_j$ of all threads is a closed subset of the product and it is called the *limit of the projective family* (or the *projective limit*). The maps $\pi_j : X \to X_j \quad \pi_j(\{x_i\}_{j\in J}) := x_j$, also called projections, are continuous and open since a basis of the topology of X consists of the subsets $\pi_j^{-1}(U_j)$, with U_j open in X_j.

Let $\{X_j, \pi_{ij}, J\}$ and $\{X'_j, \pi'_{ij}, J\}$ be projective families of topological spaces, with limit X and X', respectively. A family $\{\phi_j\}_{j\in J}$ of continuous mappings $\phi_j : X_j \to X'_j$ satisfying the coherence condition

$$\pi'_{ij} \circ \phi_j = \phi_j \circ \pi_{ij} \quad \forall j \in j, \quad i \leq j$$

is said to be a projective family of mappings. The *limit* of the projective family of mappings $\{\phi_j\}_{j\in J}$ is the map $\phi : X \to X'$ defined by

$$\phi(\{x_j\}_{j\in J}) = \{\phi_j(x_j)\}_{j\in J} .$$

The limit map ϕ is continuous and is a homeomorphism whenever each ϕ_j is a homeomorphism.

Each directed subset J_0 of J induces a projective subfamily $\{X_j, \pi_{ij}, J_0\}$. If X_0 denotes the limit of the induced projective subfamily, the map $\pi_{J_0} : X \to X_0$, defined by $\pi_{J_0}\left(\{x_j\}_{j\in J}\right) = \{x_j\}_{j\in J_0}$, is continuous and open (however π_{J_0} may not be surjective). If the directed subset J_0 is cofinal in J, then X and X_0 are homeomorphic. We recall that $J_0 \subset J$ is a cofinal subset if for every $j \in J$ there exists $j_0 \in J_0$ with $j \leq j_0$.

A projective family $\{X_j, \pi_{ij}, J\}$ is said *trivial* if the projections π_{ij} are homeomorphisms for j belonging to some cofinal subset J_0. The limit of a trivial family is homeomorphic to X_{j_0} for every $j_0 \in J_0$.

If the index set J admits a countable cofinal subset, the projective family $\{X_j, \pi_{ij}, J\}$ is called a projective sequence; in this case there exists a cofinal subset which can be identified with $\mathbf{N}$.

A projective family is said to be *surjective* if the projections π_j are surjective. This implies that all projections π_{ij} are onto. A projective family of compact spaces in which the π_{ij} are surjective maps is surjective. The same property holds for projective sequences of (not necessarily compact) spaces.

The limits of general projective families could be empty or inherit only few topological properties. More regular are limits of surjective families or limits of compact spaces: a projective limit of compact spaces is nonempty and compact. The limit of a surjective projective family of connected spaces is connected. Beware, however, that even limits of surjective projective sequences of path connected compact sets could not be path connected (see Examples 3–5 below).

We denote by $Cyl_j(X)$ for $j \in J$ the ring of the functions $f : X \to \mathbf{R}$ of the form $f = \pi_j^* f_j$, for a continuous function $f_j : X_j \to \mathbf{R}$ (i.e. f is the pullback of some $f_j \in C(X_j)$). The graduated ring $Cyl(X)$ of *cylindrical* functions on X is the union $\bigcup_{j \in J} Cyl_j(X)$. The map $\pi_j^* : C(X_j) \to Cyl(X)$ $\quad \pi_j^* f := f \circ \pi_j$ is a ring homomorphism with range $Cyl_j(X)$ and is injective if π_j is onto. Thus for surjective projective families each ring $Cyl_j(X)$ can be identified with the ring $C(X_j)$.

We say that $f : X \to \mathbf{R}$ is *locally cylindrical* if for each $x \in X$ there esists an open neighbourhood U_x of x such that the restriction $f_{|U_x}$ agrees with the pullback of some $f_j \in C(\pi_j(U_x))$. Locally cylindrical functions are continuous.

Projective limits of ordinary (i.e. finite dimensional paracompact smooth) manifolds and their differential properties are the argument of this paper. Such limits are often considered in the literature and are topological spaces which in general do not support the structure of differential manifold. Here a generalization of ordinary differential calculus is introduced appropriate to these spaces. We start with a formal definition to select a relevant class of projective limits.

Definition 1. A projective family $\{M_j, \pi_{ij}, J\}$ such that
(i) M_j are (ordinary) manifolds,
(ii) the projections $\pi_{ij} : M_j \to M_i$ are surjective submersions,
will be called a projective family of manifolds and its projective limit M a projective limit of manifolds.

To introduce elements of a differential structure on M one can use an algebraic method: the starting point is the choice of a suitable ring of functions on which vector fields are introduced as (suitable) derivations. Using algebraic definitions vector fields, differential forms, Lie brackets, Lie derivatives and other differential operators can also be defined. This is a procedure widely used also in noncommutative geometry [15,33] and on supermanifolds [23]. For a projective limit M of manifolds a natural choice appears to be the (Abelian) ring of *smooth cylindrical functions*

$$Cyl^\infty(M) := \bigcup_{j \in J} Cyl_j^\infty(M),$$

where $Cyl_j^\infty(M) := \{\pi_j^* f_j \,|\, f_j \in C^\infty(M_j)\}$ can be identified with $C^\infty(M_j)$ if the projective family is surjective. One could also use the ring of all *smooth locally cylindrical functions of M*, denoted by $Cyl_\ell^\infty(M)$. Of course, for a projective limit M of compact manifolds this ring agrees with $Cyl^\infty(M)$. The differential calculus based on $Cyl^\infty(M)$ was proposed by Ashtekar and Lewandowski in [5]. We shortly discuss this structure.

Even if differential calculus on M can be introduced on the basis of purely algebraic definitions, it is very natural to start more geometrically defining an appropriate "tangent bundle". To the projective family $\{M_j, \pi_{ij}, J\}$ we can associate the projective family of manifolds $\{TM_j, T\pi_{ij}, J\}$, whose limit we denote by TM. One easily sees that the limit map $\tau : TM \to M$ of the projections $\tau_j : TM_j \to M_j$ is continuous and onto.

We refer to (TM, τ, M) as *the tangent bundle* of M. The fibre at x, the *tangent space at* x, is the vector space $T_x M = \varprojlim_{j\in J} T_{x_j} M_j$ (which is a complete nuclear locally convex vector space by Theorem 7.4 in [45]). Notice, however, that this "bundle" does not satisfy the local triviality condition.

The tangent bundle TM plays a role very similar to the tangent bundle of a manifold. Actually, for every $f \in Cyl^\infty(M)$, $f = \pi_j^* g_j$ the *differential*

$$\mathbf{d}f : TM \to \mathbf{R}, \quad \mathbf{d}f(v_x) := d_{x_j} g_j(v_{x_j}) = (T\pi_j^* dg_j)(v_x),$$

is well defined, since the differential $d_{x_j} g_j(v_{x_j})$ does not depend on the representation $f = \pi_j^* g_j$. One easily recognizes that $\mathbf{d}f \in Cyl_j^\infty(TM)$ whenever $f \in Cyl_j^\infty(M)$ and its restriction $\mathbf{d}_x f$ on the fibre $T_x M$ is continuous and linear. Every $v_x \in T_x M$ defines a grade preserving derivation at x on $Cyl^\infty(M)$ by

$$f \leadsto L_{v_x} f := \mathbf{d}f(v_x).$$

We now assume that the projective family of manifolds is surjective and denote by D_x a grade preserving derivation at x defined on $Cyl^\infty(M)$ and by $D_{x_j} : C^\infty(M_j) \to C^\infty(M_j)$ the induced derivation at x_j, for each $j \in J$. By finite dimensionality of $T_{x_j} M_j$ there exists a (unique) $v_j \in T_{x_j} M_j$ such that D_{x_j} is the Lie derivative L_{v_j}. Since $D_{x_i} = D_{x_j} \circ \pi_{ij}^*$ for $i \le j$, one easily recognizes that the family $\{v_j\}_{j\in J}$ is a thread. Thus a $v_x \in T_x M$ is defined such that $L_{v_x} = D_x$. Therefore the following proposition holds.

Proposition 2. *Let $\{M_j, \pi_{ij}, J\}$ be a surjective projective family of manifolds with limit M. For every $x \in M$ the tangent space $T_x M$ is isomorphic with the space of all grade preserving derivations at x on $Cyl^\infty(M)$.*

Remark. The differential $\mathbf{d}f$ is well defined also if $f \in Cyl_\ell^\infty(M)$. Moreover, every tangent vector at x defines a grade preserving derivation at x on the graded ring $Cyl_\ell^\infty(M)$ and, for surjective projective families, $T_x M$ is isomorphic with the space of all grade preserving derivations at x of the ring $Cyl_\ell^\infty(M)$.

It is natural to define vector fields on M as derivations on $Cyl^\infty(M)$. Given a surjective projective family $\{M_j, \pi_{ij}, J\}$, grade preserving derivations D on $Cyl^\infty(M)$ induce on each M_j a derivation D_j and the family $\{D_j\}_{j\in J}$ satisfies the coherence condition $(\pi_{ij})_* D_j = D_i$ for $i \le j$. The Lie bracket $[D_1, D_2]$ of two grade preserving derivations is the derivation associated to the family $\{[D_{1;j}, D_{2;j}]\}_{j\in J}$. Thus grade preserving derivations on $Cyl^\infty(M)$ form a Lie algebra.

To every grade preserving derivation on $Cyl^\infty(M)$, a family $\{X_j\}_{j\in J}$ of vector fields with $\pi_{ij}^* X_j = X_i$ for $j \le i$ is associated and a section $X : M \to TM$ is defined by the limit of these vector fields. We remark that one can recover the fields X_j by X since $T\pi_j \circ X : M \to TM_j$ depends only on the components in M_j and that this property characterizes limits of vector fields. The set of these limits is a Lie algebra with $[X, Y] :=$

$\varprojlim_{j \in J}[X_j, Y_j]$. Conversely, to every limit of vector fields a grade preserving derivation D on $Cyl^{\infty}(M)$ is associated (and Lie brackets are conserved). Thus we get the next proposition.

Proposition 3. *Let M be the limit of a surjective projective family of manifolds $\{M_j, \pi_{ij}, J\}$. Grade preserving derivations on $Cyl^{\infty}(M)$ and projective limits of vector fields are isomorphic Lie algebras.*

We remark that the objects and the isomorphism in the above proposition depend on the given projective family. However, if one takes in J a cofinal subset J_0, the ring of cylindrical functions does not change, while one has to consider derivations conserving the grading just for labels in J_0.

One could consider as vector fields on M the limits of vector fields arising by cofinal subsets J_0 of J. However, this set of fields could not admit a Lie bracket. A good Lie bracket is defined if one considers only cofinal subsets of the type $\{j \in J \mid j \geq j_0\}$ for a given $j_0 \in J$, as in [5].

Differential cylindrical forms are defined in an analogous way as cylindrical functions, considering the pullback on M of differential forms on the M_j. Usual differential operations as Lie derivatives, exterior derivative, etc. and cylindrical cohomology are estabilished.

Here we introduce some examples of projective limits of manifolds, some of which we shall use as toy model in the sequel.

Example 1. For a projective family $\{G_\alpha, \pi_{\alpha\beta}, A\}$, where G_α are Lie groups and the projections are homomorphisms onto, the limit of G is a topological group and the projections π_α are homomorphisms. Notable examples are compact groups: it is well known that every compact group is the projective limit of a family of compact Lie groups [48].

As Lie groups are parallelizable, the tangent space at the unit e of G is $\mathfrak{g} := T_e G = \varprojlim_{\alpha \in A} T_{e_\alpha} G_\alpha$. Then $TG = \varprojlim_{\alpha \in A} TG_\alpha = \varprojlim_{\alpha \in A}(G_\alpha \times \mathfrak{g}_\alpha) = G \times \mathfrak{g}$. An exponential map $\exp : \mathfrak{g} \to G$ can also be defined as the limit of the family of maps $\{\exp_\alpha\}_{\alpha \in A}$. This exponential map is continuous, but not open in general.

Every neighbourhood U_e of the unit e of G contains the kernel H_α of the projections π_α, so that G does admit small subgroups if the normal subgroups H_α are not definitively trivial (i.e. if the projective family is not trivial). Therefore, the projective limit of a nontrivial projective family of Lie groups cannot be an ordinary Lie group by the Yamabe Theorem [49]. However, a projective limit of ordinary Lie groups may be an infinite-dimensional Lie group. As a simple example we recall that $\mathbf{R}^{\mathbf{N}}$, the space of real sequences with the product topology, is a Fréchet space, hence an Abelian Lie group and it is the projective limit of the Abelian Lie groups $\mathbf{R}^d$, $d \in \mathbf{N}$. In this case Yamabe Theorem does not apply: $\mathbf{R}^{\mathbf{N}}$ admits indeed small subgroups.

We stress that projective limits of a nontrivial family of compact Lie groups cannot be Lie groups (as already mentioned in the introduction) since in this case the limits are compact groups and Yamabe theorem does apply.

Example 2. Now we give an example of a projective sequence of manifolds whose projective limit is a manifold modelled on a Fréchet vector space [35].

The set $J^k(M, N)$ of k-jets of smooth mappings between manifolds M and N, with dimension m and n, respectively, is an ordinary affine fibre bundle over $M \times N$ with fibre at (x, y) the linear space $P^k(m, n) := \prod_{j=1}^{k} L_s^j(\mathbf{R}^m, \mathbf{R}^n)$, where $L_s^j(\mathbf{R}^m, \mathbf{R}^n)$ denotes the space of j-linear symmetric mappings $\mathbf{R}^m \to \mathbf{R}^n$.

There are natural projections $\pi_{h,k} : J^k(M, N) \to J^h(M, N)$ for $h < k$, which in local charts are truncations of Taylor polynomials up to order h. As the projections satisfy the coherence property, the family $\{J^k(M, N), \pi_{h,k}, \mathbf{N}\}$ is a projective sequence of manifolds (actually, of affine bundles).

The projective limit $J^\infty(M, N)$ of this sequence consists of the Taylor expansions of smooth mappings and is a manifold modelled on a nuclear Fréchet space [35]. Actually, the limit map $J^\infty(M, N) \to M \times N$ of the projective family of projection maps is an affine fibre bundle projection with fibre on the nuclear Fréchet space $P^\infty(m, n)$ of all symmetric formal power series, i.e. the projective limit of the spaces $P^k(m, n)$ (and the product of the spaces $L_s^j(\mathbf{R}^m, \mathbf{R}^n)$, $i \geq 1$).

Example 3. A wide class of projective families of manifolds is obtained giving just a manifold X and a map $\phi : X \to X$ which is local diffeomorphism onto X. The associated projective sequence is $\{M_n, \pi_{n,m}, \mathbf{N}\}$ where $M_n = X$ and $\pi_{n,m} := \phi^{m-n}$ for $n < m$. Projective limits of this type arise in the theory of dynamical systems [47].

Since at any point $x_m \in M_m$ the tangent map $T_{x_m}\pi_{m,n} : T_{x_m}M_m \to T_{x_n}M_n$ is a linear isomorphism, the projective sequence of tangent spaces is trivial, so that the tangent space at $x \in M$ is $T_x M = \varprojlim_{n\in\mathbf{N}} T_{x_n} M_m \simeq T_{x_1} X$. As every projective sequence of manifolds is surjective, cylindrical maps are identified with smooth functions defined on some M_n.

A simple but typical example is the *p-adic solenoid* Σ_p, $p \in \mathbf{N}$, $p > 1$, constructed as above with $X = S^1$ and $\phi : S^1 \to S^1 \quad \phi(z) := z^p$ (see [17,21]). The projections are group homomorphisms and coverings.

It is well known that Σ_p is isomorphic with the compact Abelian group $(\mathbf{R} \times \Delta_p)/\mathbf{B}$. Here Δ_p is the group of p-adic integers, i.e. of formal series $\mathbf{x} = x_0 + x_1 p + \cdots + x_k p^k + \cdots$ where the coefficients are integers satisfying the inequalities $0 \leq x_k < p$, $k = 0, 1, 2, \ldots$ and $\mathbf{B}$ denotes the subgroup generated by the element $(1, \mathbf{u})$, with $\mathbf{u} \in \Delta_p$ defined by $u_k = \delta_{0,k}$ for $k = 0, 1, \ldots$ We recall that Δ_p is the projective limit of the sequence of discrete groups $\mathbf{Z}/p^n\mathbf{Z}$; therefore it is a Cantor group, i.e. an uncountable compact Abelian group which is a perfect totally disconnected space.

An isomorphism with Σ_p can be constructed as follows. Let $\chi_n : \mathbf{R} \times \Delta_p \to S^1$ be the epimorphism defined by

$$\chi_n(t, \mathbf{x}) = \exp\left(\frac{2\pi \mathrm{i}}{p^n}(t - (x_0 + x_1 p + \cdots + x_{n-1} p^{n-1}))\right).$$

Since $(\chi_m)^{p^{m-n}} = \chi_n$ for $n < m$, the family $\{\chi_n, \mathbf{N}\}$ is a projective family of maps. Therefore the limit map χ is defined and is a group homomorphism of $\mathbf{R} \times \Delta_p$ onto Σ_p. The kernel of χ is the group $\mathbf{B}$ so that χ quotients to the wanted isomorphism $\tilde{\chi} : (\mathbf{R} \times \Delta_p)/\mathbf{B} \to \Sigma_p$. The p-adic solenoid Σ_p is a connected compact group but it is not arcwise connected, not even locally connected. The path components are precisely the images of the continuous homomorphism $\eta_{\mathbf{x}} : \mathbf{R} \to \Sigma_p$ defined by $\eta_{\mathbf{x}}(t) := [(t, \mathbf{x})]$, with dense image and kernel zero. Moreover, the projection $\mathbf{R} \times \Delta_p \to \Delta_p$ quotients to a (not continuous) group epimorphism $\Sigma_p \to \Delta_p/\mathbf{uZ}$, whose fibres are exactly the path components. Thus there are uncountably many path components, classified by the Cantor group $\Delta_p/\mathbf{uZ}$, each dense (see Remarque 1 in [16]).

Example 4. It is well-known that $\mathbf{R}$ is the universal covering of S^1 and that $\pi^1(S^1) = \mathbf{Z}$. For every integer $p \in \mathbf{N}$, consider the subgroups $G_p = p\mathbf{Z}$ of $\mathbf{Z}$ and the manifolds $M_p := \mathbf{R}/G_p$, all diffeomorphic to S^1. If on $\mathbf{N}$ the ordering is given by $p \preceq q$ if p divides q (so that $G_q \lhd G_p$), the quotient map $\pi_{qp} : M_q \to M_p$ is defined for $p \preceq q$. So we have a projective surjective family of finite sheeted coverings of S^1, which are group epimorphisms. The limit Σ_∞ of this family is a compact connected Abelian group and projects on Σ_p for every $p \in \mathbf{N}$. Therefore, Σ_∞ admits uncountable many path components, each dense.

Example 5. The *universal laminated surfaces* have been introduced and studied by Nag and Sullivan ([41,47] and also [11]) in their investigations on the system of Teichmüller spaces of Riemann surfaces of different genera. The relevance of these spaces in path integral quantization of nonperturbative String Theory was discussed in [40]. For a closed (i.e. compact, connected, without border) Riemann surface X_g of genus g, equipped with a base point $\star$, the authors considered the set J_g of all homotopy classes of finite sheeted unbranched pointed covering maps $\alpha : X_\alpha \to X_g$, where X_α is a closed Riemann surface. This set is directed under the partial ordering given by factorization, i.e. $\alpha \preceq \beta$ if there is a commuting triangle of covering maps $\beta = \alpha \circ \theta$. The ordered set J_g has a minimum ι corresponding to the identity map on X_g. To every α a monomorphism $\pi_1(\alpha) : \pi_1(X_\alpha, \star) \to \pi_1(X_g, \star)$ is associated. Thus $\alpha \preceq \beta$ if and only if $\mathrm{Im}(\pi_1(\beta)) \subset \mathrm{Im}(\pi_1(\alpha))$.

If a universal covering $(X, \star)$ over $(X_g, \star)$ is fixed, $\pi_1(X_g, \star)$ is identified with the group G (acting on X) of the deck transformations of X_g and $\mathrm{Im}(\pi_1(\alpha))$ with a subgroup G_α. Thus $X_g = X/G$ and a closed Riemann surface $S_\alpha := X/G_\alpha$ is constructed for each α. For $\alpha \preceq \beta$ the projection $\pi_{\alpha,\beta} : S_\beta \to S_\alpha$ is defined in the obvious way, so one has a projective family $\{S_\alpha, \pi_{\alpha\beta}, J_g\}$ of coverings of X_g. Utilizing only normal subgroups of G would give a cofinal projective subfamily.

If $g = 1$, X_g is a torus, $(\mathbf{C}, \star)$ is a universal covering, $\mathbf{Z} \oplus \mathbf{Z}$ is the fundamental group and all coverings are also tori. The projective limit is called the *universal Euclidean lamination* E_∞. The projective family of tori defining E_∞ consists of the quotients $\mathbf{C}/(p\mathbf{Z} \oplus q\mathbf{Z})$, $p, q \in \mathbf{N}$. Hence $E_\infty = \Sigma_\infty \times \Sigma_\infty$.

Each surface X_g of genus ≥ 2 has the Poincaré hyperbolic half-plane as universal cover. The limit H_∞ projects on surfaces of every genus ≥ 2. It is therefore called the *universal hyperbolic lamination*.

3. $\mathcal{C}^\infty$-spaces

We present here the class of $\mathcal{C}^\infty$ spaces introduced by Fröhlicher and Kriegl in [19]. This is a very large category containing Fréchet manifolds and has nice mathematical properties: the set of all $\mathcal{C}^\infty$ functions between each pair of $\mathcal{C}^\infty$ spaces has a canonical structure of $\mathcal{C}^\infty$ space (Cartesian closedness of the category); moreover the $\mathcal{C}^\infty$ category is closed with respect to inductive and projective limits. In particular the last property makes the proposal of Fröhlicher and Kriegl particularly interesting for us. Previous attempts to generalize differential calculus according to similar ideas, are the differential spaces of Smith [46] and Chen [14]. As a consequence of Boman Theorem, their approach is essentially equivalent to that of $\mathcal{C}^\infty$ spaces.

The idea in $\mathcal{C}^\infty$ spaces is to define a differential structure on a set X by a family $\mathcal{C}$ of curves $c : \mathbf{R} \to X$ and a family $\mathcal{S}$ of functions $f : X \to \mathbf{R}$ with the property that $\mathcal{C}$ and $\mathcal{S}$ determine each other by the conditions:

$$\mathcal{S} = \{f : X \to \mathbf{R} | f \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})\ \forall c \in \mathcal{C}\},$$

$$\mathcal{C} = \{c : \mathbf{R} \to X | f \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})\ \forall f \in \mathcal{S}\}.$$

The elements of $\mathcal{C}$ are called *structure curves* or $\mathcal{C}^\infty$ curves (or simply curves), those of $\mathcal{S}$ the *structure functions* or $\mathcal{C}^\infty$ functions. The pair $(\mathcal{C}, \mathcal{S})$ is called a $\mathcal{C}^\infty$-structure on X and the triple $(X, \mathcal{C}, \mathcal{S})$ is said a *$\mathcal{C}^\infty$-space.*

A set C of curves in X is *generating* for $(\mathcal{C}, \mathcal{S})$ if $\mathcal{S} = \{f : X \to \mathbf{R} \mid f \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})\ \forall c \in C\}$. Analogously, a set of functions S on X is generating for $(\mathcal{C}, \mathcal{S})$ if $\mathcal{C} = \{c : \mathbf{R} \to X \mid f \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})\ \forall f \in S\}$.

A $\mathcal{C}^\infty$ map between $\mathcal{C}^\infty$ spaces $(X_1, \mathcal{C}_1, \mathcal{S}_1)$ and $(X_2, \mathcal{C}_2, \mathcal{S}_2)$ is a map $g : X_1 \to X_2$ satisfying one of the following equivalent conditions:

$$g \circ c \in \mathcal{C}_2 \quad \forall c \in \mathcal{C}_1,$$

$$f \circ g \in \mathcal{S}_1 \quad \forall f \in \mathcal{S}_2,$$

$$f \circ g \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}) \quad \forall f \in \mathcal{S}_2,\ \forall c \in \mathcal{C}_1.$$

The set of all $\mathcal{C}^\infty$ maps from X_1 to X_2 is denoted $\mathcal{C}^\infty(X_1, X_2)$.

On an ordinary manifold M a $\mathcal{C}^\infty$ structure $(\mathcal{C}, \mathcal{S})$ is given where $\mathcal{C} := \mathcal{C}^\infty(\mathbf{R}, M)$ and $\mathcal{S} := \mathcal{C}^\infty(M, \mathbf{R})$. The set of $\mathcal{C}^\infty$ maps between two manifolds M and N is precisely the set $\mathcal{C}^\infty(M, N)$ of the smooth functions. This is a consequence of the Boman Theorem [12]. For every $\mathcal{C}^\infty$ space X, the set $\mathcal{C}$ of structure curves is precisely $\mathcal{C}^\infty(\mathbf{R}, X)$, while the set $\mathcal{S}$ of structure functions is $\mathcal{C}^\infty(X, \mathbf{R})$, briefly denoted by $\mathcal{C}^\infty(X)$.

A $\mathcal{C}^\infty$ structure is defined on products $\prod_{t \in T} X_t$, admitting $\prod_{t \in T} \mathcal{C}_t$ as a set of structure curves, where $\mathcal{C}_t$ denotes the set of structure curves in X_t, for $t \in T$. For every pair X_1, X_2 of $\mathcal{C}^\infty$ spaces, the set $\mathcal{C}^\infty(X_1, X_2)$ a canonical $\mathcal{C}^\infty$ structure is given, setting

$$\mathcal{C}^\infty(\mathbf{R}, \mathcal{C}^\infty(X_1, X_2)) := \mathcal{C}^\infty(\mathbf{R} \times X_1, X_2)\ .$$

For $\mathcal{C}^\infty$ spaces X_1, X_2, X_3, one gets the canonical isomorphism

$$\mathcal{C}^\infty(X_1, \mathcal{C}^\infty(X_2, X_3)) \simeq \mathcal{C}^\infty(X_1 \times X_2, X_3).$$

This amounts to the Cartesian closedness of the $\mathcal{C}^\infty$ category, which so appears as a general scenario for Cartesian closed categories of spaces supporting a differential calculus and containing ordinary manifolds; for the proof of Cartesian closedness, see 1.4.3 in [19]. However, one encounters serious difficulties to define a good tangent space and a differential of $\mathcal{C}^\infty$ maps, for a quite general $\mathcal{C}^\infty$ space. Of course, one could proceed as in ordinary manifolds to obtain the kinematical tangent space according to the following definition.

Definition 4. Two curves c_1, c_2 of a $\mathcal{C}^\infty$ space X are said to be *tangent* at $x \in X$ if $c_1(0) = c_2(0) = x$ and

$$(f \dot{\circ} c_1)(0) = (f \dot{\circ} c_2)(0) \quad \forall f \in \mathcal{S}.$$

The equivalence class $[c]_x$ of c is called the velocity vector of c at x. The set of all velocity vectors of curves at x is the kinematical tangent space at x, denoted by $\mathcal{T}_x X$.

In spite of its name, $\mathcal{T}_x X$ can fail to have the full structure of linear space. A simple example where $\mathcal{T}_x X$ is not linear is the following. Take $X = X_1 \cup X_2$ where X_1 and X_2 are orthogonal real lines at 0 in $\mathbf{R}^2$. Structure curves in X are smooth curves in $\mathbf{R}^2$ with values in X. The kinematical tangent space at 0 is identified with X itself, so it is not linear.

Every $f \in \mathcal{S}$ does admit a *kinematical differential* at x defined by

$$\delta_x f : \mathcal{T}_x X \to \mathbf{R} \qquad v_x \rightsquigarrow \delta_x f(v_x) := (f \dot{\circ} c)(0), \quad c \in v_x.$$

On $\mathcal{T}X$, the disjoint union $\bigcup_{x \in X} \mathcal{T}_x X$, a surjective map $\tau : \mathcal{T}X \to X$ is defined by $\tau(v_x) := x$. If one assumes $\{\delta f \mid f \in \mathcal{S}\} \cup \{\tau^* f \mid f \in \mathcal{S}\}$ as a generating set of functions for a $\mathcal{C}^\infty$ structure on $\mathcal{T}X$, the map $\tau : \mathcal{T}X \to X$ is a $\mathcal{C}^\infty$ map. We refer to $(\mathcal{T}X, \tau, X)$ as the *kinematical tangent bundle*. In particular, if X is an ordinary manifold, then $\mathcal{T}X$ is just the usual tangent bundle TX and $\delta_x f$ the ordinary differential.

Even if the kinematical tangent bundle appears as a natural object, there are some contexts where another tangent space naturally arises: in the case of a projective limit of manifolds $M = \varprojlim_{j \in J} M_j$, one should assure that good $\mathcal{C}^\infty$ functions admit a differential defined on $TM = \varprojlim_{j \in J} TM_j$. A right balance between $\mathcal{C}^\infty$ curves and $\mathcal{C}^\infty$ functions appears necessary to obtain good tangent spaces and good differentials for $\mathcal{C}^\infty$ functions. Actually, in [19] the general theory of $\mathcal{C}^\infty$ spaces is not fully developed. The main of the book concerns $\mathcal{C}^\infty$ calculus for a particular class of locally convex vector spaces, called *convenient* vector spaces by the authors, where straight lines assure a richness of curves to get nice differential calculus.

In a locally convex vector space E the structure curve set $\mathcal{C}$ is the family of infinitely many differentiable curves, where a curve $c : \mathbf{R} \to E$ is differentiable if the derivate $\dot{c}(t) := \lim_{h \to 0}(1/h)(c(t+h) - c(t))$ exists for every $t \in \mathbf{R}$ and the map $t \rightsquigarrow \dot{c}(t)$ is continuous. The set $\mathcal{C}$ does not really depend on the locally convex topology of E, but only on the system of its bounded sets, so that $\mathcal{C}^\infty$ functions are not necessarily continuous. This cannot be avoided in any calculus, if Cartesian closedness is wanted: actually the evaluation $E \times E' \to \mathbf{R}, \quad (x, \ell) \rightsquigarrow \ell(x) \quad x \in E, \ell \in E'$ (the dual space) has to be a $\mathcal{C}^\infty$ function but it is jointly continuous if and only if E is normable.

Every continuous linear functional on E is a $\mathcal{C}^\infty$ function. A separated locally convex vector space E is called a *convenient vector space* whenever its dual space E' is a generating set of functions for the $\mathcal{C}^\infty$ structure of E. The name refers to the fact that this class of spaces is Cartesian closed and supports a good calculus.

The kinematical tangent space at $x \in E$, for a convenient vector space E, is precisely E. For every $f \in \mathcal{C}^\infty(E)$ the kinematical differential $\delta_x f$ at $x \in E$ is a continuous linear map and agrees with the usual differential $d_x f$ defined by

$$d_x f(v) = \lim_{t\to 0}(1/t)(f(x+tv) - f(x)) .$$

Differential calculus in convenient vector spaces is based on the following theorem (see Proposition 4.4.9 of [19]).

Proposition 5. *Let E be a convenient vector space and $f \in \mathcal{C}^\infty(E, \mathbf{R})$. Then the differential operator*

$$d : \mathcal{C}^\infty(E, \mathbf{R}) \to \mathcal{C}^\infty(E \times E, \mathbf{R}), \quad f \rightsquigarrow df$$

is linear and $\mathcal{C}^\infty$.

As a consequence every $\mathcal{C}^\infty$ function admits iterated differentials of any order.

Fréchet spaces are convenient vector spaces and the $\mathcal{C}^\infty$ calculus agrees with the C_c^∞ calculus (we refer the reader to Appendix A, where a version of Boman Theorem for Fréchet spaces is given). Thus each $\mathcal{C}^\infty$ function f on a Fréchet space E is continuous and its differential in the $\mathcal{C}^\infty$ calculus agrees with the usual differential df in the C_c^∞ calculus.

The theory of convenient infinite-dimensional manifolds has been approached in [27], where some manifolds suitable for Algebraic Topology are discussed and in the book [28] devoted to Global Analysis. A similar, but different, philosophy has been assumed by Michor in his pioniering work [35]. If M is a manifold modelled on Fréchet spaces with C_c^∞ transition functions, the C_c^∞ functions on M are precisely the $\mathcal{C}^\infty$ functions in the $\mathcal{C}^\infty$ structure generated by C_c^∞ curves and the C_c^∞ curves agree with the $\mathcal{C}^\infty$ curves provided the local model admits bump functions. This is the case, for instance, of nuclear Fréchet spaces [32].

We are interested to consider $\mathcal{C}^\infty$ spaces which are not manifolds in any sense, as in the following examples.

Example 6. The main examples of $\mathcal{C}^\infty$ spaces are manifolds. But in foliation theory differential objects arise that are not manifolds. For generalities on foliations see [13,37].

We recall that a separable, locally compact metrizable space M is said to be a d-dimensional foliated space (or a lamination) if it admits a cover by open subsets U_i (the charts) and homeomorphisms

$$\varphi_i : U_i \to D_i \times T_i,$$

where D_i is open in $\mathbf{R}^d$ and T_i any metric space. The overlap maps are required to be of the form

$$(\varphi_j \circ \varphi_i^{-1})(z, t) = (\lambda_{ji}(z, t), \tau_{ji}(t))$$

and of class C_l^∞ : this means that λ_{ji} is smooth in the variable z, with all partial derivatives continuous in both variables. Sets of the type $\varphi_i^{-1}(D_i \times \{t\})$ glue together to form d-dimensional manifolds, whose connected components are called *leaves*.

A C_l^∞ calculus is accordingly defined: a map $f : M \to N$ between foliated spaces is said to be of class C_l^∞ if it is continuous, takes leaves to leaves and, for every pair of charts φ in M and ψ in N, the local expression $\psi \circ f \circ \varphi^{-1}$ is of class C_l^∞. The inclusion of leaves in M cannot be a homeomorphism; it is a homeomorphism with respect to the "leaf topology", obtained by putting on the transversal sets T_i the discrete topology. The foliated tangent bundle $T_l M$ is defined as the disjoint union of the tangent bundles of the leaves and admits a natural structure of foliated space defined in an obvious way.

A natural $\mathcal{C}^\infty$ structure on M arises, assuming $\mathcal{C}$ to be the set $C_l^\infty(\mathbf{R}, M)$ of all C_l^∞ curves. The range of a C_l^∞ curve is contained in a leaf and is a smooth curve in this leaf. Accordingly, $\mathcal{C}^\infty$ functions are just families $\{f_\ell\}$ of smooth real functions, one for each leaf ℓ. Thus $\mathcal{C}^\infty$ functions may not contain informations on the transversal topology and $\mathcal{C}^\infty(M, \mathbf{R})$ agrees with $C_l^\infty(M, \mathbf{R})$ only if the topology on M is the leaf topology. The kinematical tangent bundle $\mathcal{T}M$ coincides as $\mathcal{C}^\infty$ space with the foliated tangent bundle and every $\mathcal{C}^\infty$ function admits iterated differentials (along the leaves). C_l^∞ maps are precisely the $\mathcal{C}^\infty$ functions whose iterated differentials are continuous. This result is a trivial extension of Boman Theorem to d-dimensional foliated spaces.

Examples of foliated spaces where $C_l^\infty(M) \subset \mathcal{C}^\infty(M)$ strictly are the spaces Σ_p, Σ_∞, E_∞ and H_∞ introduced in Section 2. Here we shortly give their foliated atlases and we refer to Section 2 for notations.

A two-charts foliated atlas for Σ_p is given by restricting the quotient map $\mathbf{R} \times \Delta_p \to \Sigma_p$, respectively, to $(0, 1) \times \Delta_p$ and $(-1/2, 1/2) \times \Delta_p$. The leaves of Σ_p are precisely the images of the homomorphisms $\eta_{\mathbf{x}}$, so they are dense. Hence C_l^∞ functions are univocally defined by their restriction to any leaf.

Foliated atlases can be constructed in a general way for the spaces Σ_∞, E_∞ and H_∞, owing to the fact that they are limits of covering manifolds. As an example, we give a foliated atlas for the universal hyperbolic lamination H_∞. For a pointed Riemann surface $(X_g, \star)$ with $g \geq 2$, choose a universal cover $(X, \star)$. Fix an open subset U of X_g such that U is the image, by the canonical projection $X \to X_g$, of an open subset of the form $B.G$, where B is an open disk contained in a fundamental domain in X for the action of $G = \pi^1(X_g)$, denoted by the dot. By the coherence condition we see that for each normal covering surface $S_\alpha = X/G_\alpha$, the inverse image of U by the projection $\pi_{\alpha,\iota}$: $S_\alpha \to X_g$ is $(B.G)/G_\alpha \simeq B \times G/G_\alpha$. The groups $C_\alpha := G/G_\alpha$ are finite and form a projective family of groups, whose limit $\mathfrak{C}$ is a Cantor group. Thus the inverse image $\pi_\iota^{-1}(U)$ in H_∞ is the inverse limit of the family $\{B \times C_\alpha\}$, hence is homeomorphic to $B \times \mathfrak{C}$; varying U, we obtain a foliated atlas. Also in this context, there are uncountably many path components, the leaves, parametrized by the Cantor set $\mathfrak{C}$, each dense. Foliated atlases in Σ_∞ and E_∞ are obtained in an analogous way, by means of the corresponding universal covering space. There are uncountably many leaves, parametrized by a Cantor set, each dense.

In this paper we just consider real differential structure on universal laminations E_∞ and H_∞. More appropriately, complex structures have been defined on universal laminations in [41], in which each leaf of H_∞ is identified with the Poincaré hyperbolic half-plane and leaves of E_∞ with the complex plane. The Teichmüller space of H_∞ is a completion of the inductive limit of the Teichmüller spaces of the surfaces S_α. This Teichmüller space is expected to play a relevant role in path quantization of String Theory.

Example 7. Loop groups are relevant objects in the context of the loop representation of Yang–Mills Theories and Gravitation [20,44]. Different notions of loop group are given in literature and not all compatible with a Lie group structure. For instance, the loop group considered in [10] is embedded in an infinite-dimensional Lie group, the special extended loop group, but it does not contain any nontrivial one-parameter subgroup.

An interesting example of $\mathcal{C}^\infty$ structure has been recently proposed for loop groups (see [9,31]). Let $P(B, G)$ be a principal bundle with G a compact connected Lie group and B a connected manifold. Two principal bundles $P_1(B, G)$ and $P_2(B, G)$ are said to be gauge isomorphic if there exists a bundle isomorphism $\varphi : P_1 \to P_2$ such that $\varphi(xg) = \varphi(x)g$, for every $x \in P_1$ and $g \in G$. We denote by $\mathcal{G}$ the group of gauge automorphisms of $P(B, G)$. By a (parametrized) path in B we mean a continuous map $\alpha : [0, 1] \to B$ which is piecewise smooth, i.e. the interval $[0, 1]$ can be decomposed as finite union of subintervals $[s_i, s_{i+1}]$ on which α is smooth. A path α is said to be a loop if $\alpha(0) = \alpha(1)$; the loop $s \rightsquigarrow \alpha(1-s)$ is denoted α^{-1}.

On the set of loops based on $\star$, a composition is defined by

$$(\alpha \circ \beta)(s) = \begin{cases} \alpha(2s), & s \in [0, 1/2], \\ \beta(2s-1), & s \in [1/2, 1]. \end{cases}$$

The main tool in the loop representation of Yang–Mills Theories is however the loop group $\mathcal{L}_\star$ consisting of the equivalence classes of loops based on $\star$, with respect to the relation:

$$\alpha \sim \beta \quad \text{if } H_A(\alpha) = H_A(\beta) \tag{1}$$

for every connection A on $P(B, G)$, see [31]. Here $H_A(\alpha)$ denotes the holonomy of A along α, defined as follows. The parallel transport along α of the connection A is an equivariant automorphism $\mathcal{P}_\alpha^A$ of the fibre $P_\star$ over the point $\star$; if a point $x_0 \in P_\star$ is fixed, this automorphism is identified with the element $H_A(\alpha)$ of the structure group G satisfying $\mathcal{P}_\alpha^A(x_0)H_A(\alpha) = x_0$. We recall that $H_A(\alpha \circ \beta) = H_A(\alpha)\, H_A(\beta)$ and $H_A(\alpha^{-1}) = H_A(\alpha)^{-1}$. If A_1 and A_2

are gauge equivalent, their holonomy maps are gauge equivalent, i.e. there exists $g \in G$ such that $H_{A_1}(\alpha) = gH_{A_2}(\alpha)g^{-1}$ for every loop α.

The set $\mathcal{L}_\star$ becomes a group if its product is defined by $[\alpha] \circ [\beta] := [\alpha \circ \beta]$; the quotient map $H_A : \mathcal{L}_\star \to G$, $H_A([\alpha]) = H_A(\alpha)$ is a homomorphism of groups, called the holonomy map of the connection A.

A $\mathcal{C}^\infty$-structure on $\mathcal{L}_\star$ is generated by the set of curves

$$\{c : \mathbf{R} \to \mathcal{L}_\star,\ c(t) = [\alpha_t]\},$$

where $t \rightsquigarrow \alpha_t$ is a homotopy of loops, i.e. the map

$$h : \mathbf{R} \times [0, 1] \to B, \quad h(t, s) := \alpha_t(s),$$

is continuous and there exists a partition $0 = s_1 < s_2 < \cdots < s_k = 1$ of the unit interval such that, for every i,

$$h : \mathbf{R} \times (s_i, s_{i+1}) \to B$$

is smooth. With respect this $\mathcal{C}^\infty$ structure the group operations in $\mathcal{L}_\star$ are $\mathcal{C}^\infty$.

This notion of $\mathcal{C}^\infty$ map is essential to characterize holonomy maps of smooth connections in the space $\mathrm{Hom}(\mathcal{L}_\star, G)$ of group homomorphisms. The holonomy map H_A associated to a smooth connection A is a $\mathcal{C}^\infty$ map: for every curve in $\mathcal{L}_\star$, the curve $\mathbf{R} \ni t \rightsquigarrow H_A(\alpha_t) \in G$ is smooth since it is obtained (locally) as solution of a vector field on G depending smoothly on the parameter t (see II.3 in [25]). The correspondence $H : A \rightsquigarrow H_A$ was widely studied (see [31] and the bibliography therein). We summarize their results in the next proposition, where by $\mathrm{Hom}^\infty(\mathcal{L}_\star, G)$ we denote the space of $\mathcal{C}^\infty$ homomorphisms of $\mathcal{L}_\star$ in G.

Proposition 6. *The map H defines a one to one correspondence (up to gauge equivalence) between smooth connections on smooth G-principal bundles on B and the elements of* $\mathrm{Hom}^\infty(\mathcal{L}_\star, G)$.

In [31] analogous $\mathcal{C}^\infty$ structures are considered on path bundles and generalized path principal bundles.

4. Products of manifolds

Here we consider a product space $M = \prod_{t \in T} M_t$ of ordinary manifolds M_t, where the cardinality of the index set T is assumed to be $\leq 2^{\aleph_0}$. M is the limit of the projective surjective family of manifolds $\{M_j, \pi_{ij}, J\}$, where J denotes the directed set of all finite subsets j of T and $M_j = \prod_{t \in j} M_t$. For a subset $S \subset T$ we denote π_S the projection of M onto $\prod_{t \in S} M_t$.

We recall that a canonical $\mathcal{C}^\infty$ structure is given on M, where the set $\mathcal{C}$ of structure curves consists of families $c = \{c_t\}_{t \in T}$ of smooth curves c_t in M_t. The ring $Cyl^\infty(M)$ is in general only a generating set of functions: f.i. functions in $Cyl^\infty_\ell(M)$ are $\mathcal{C}^\infty$. We will consider also countably cylindrical $\mathcal{C}^\infty$ functions, i.e. functions $f = \pi^*_{T_0} f_{T_0}$ which are the pullback of a C^∞ function $f_{T_0} : \prod_{t \in T_0} M_t \to \mathbf{R}$, for a countable $T_0 \subset T$. Countably cylindrical C^∞ functions on M which are not cylindrical do exist; we are indebted to A. Kriegl for the following example of a locally cylindrical function on $\mathbf{R}^\mathbf{N}$ and for the next proposition.

Example 8. Let $h \in C^\infty(\mathbf{R}, \mathbf{R})$, supp $h \subset [-1/2, 1/2]$, $h(0) = 1$; the function $f : \mathbf{R}^\mathbf{N} \to \mathbf{R}$, $f(x) := \sum_{n=0}^{\infty} h(x_0 - n)\, x_n$ is $C^\infty(\mathbf{R}^\mathbf{N})$, locally cylindrical but not cylindrical.

Proposition 7. *Every $f \in C^\infty(\mathbf{R}^\mathbf{N})$ is locally cylindrical.*

Proof. By Theorem A.2 in Appendix A, every C^∞ function f on the Fréchet space $\mathbf{R}^\mathbf{N}$ is a C_c^∞ function, hence $df : \mathbf{R}^\mathbf{N} \times \mathbf{R}^\mathbf{N} \to \mathbf{R}$ is continuous. Let now $U \times V$ be a connected open subset of $\mathbf{R}^\mathbf{N} \times \mathbf{R}^\mathbf{N}$ such that $|df(x, v)| < 1$ for every $(x, v) \in U \times V$. We can assume that $V = \prod_{n \in \mathbf{N}} V_n$ where V_n are open subsets of $\mathbf{R}$ which equal $\mathbf{R}$, except for a finite set $\mathbf{N}_0$ of indices.

Using linearity of df in the second variable one proves that $df(x, v) = 0$ if $(x, v) \in U \times V$ and $v_n = 0$ for $n \in \mathbf{N}_0$. For $x, y \in U$ and a smooth curve $c \in \mathbf{R}^\mathbf{N}$ joining x and y, one has

$$f(y) = f(x) + \int_0^1 df\,(c(s), \dot{c}(s))\, \mathrm{d}s.$$

If $x_n = y_n$ for every $n \in \mathbf{N}_0$, one can choose c in U such that $\dot{c}(s)_n = 0\ \forall n \in \mathbf{N}_0$ to get $f(x) = f(y)$. □

In this case the restriction to cylindrical smooth functions appears unnecessary: using a standard notion of derivative in Fréchet spaces one obtains the wider class Cyl_ℓ^∞ of smooth functions.

On a product of manifolds $M = \prod_{t \in T} M_t$ we can construct analogous examples of C^∞ functions which are locally cylindrical, but not cylindrical, if at least one of the factors M_t is not compact. We have even simple examples of C^∞ functions which are not continuous, hence not ever locally cylindrical. Let M be $(\mathbf{2} \times S)^\mathbf{N}$ where S is an ordinary manifold and $\mathbf{2}$ denotes the space consisting of two elements. So $M = \mathbf{2}^\mathbf{N} \times S^\mathbf{N}$ and C^∞ curves in M are maps $s \rightsquigarrow \{\xi_n \times c_n(s)\}_{n \in \mathbf{N}}$, where c_n is a smooth curve in S for every n. Choose any noncontinuous function h on $\mathbf{2}^\mathbf{N}$. The function f on M defined by $f(\xi, s) = h(\xi)$, $\xi \in \mathbf{2}^\mathbf{N}$, $s \in S$, is C^∞ but not continuous.

The main result in this section is that C^∞ functions on a reasonable product M of manifolds are continuous and locally cylindrical, hence cylindrical whenever M is compact. First we prove that, in a product of connected manifolds, every C^∞ function is countably cylindrical. We need some lemmas.

Lemma 8. *Let (M, g) be an ordinary connected Riemannian manifold, d_g the metric distance, $\{x_n\}_{n \in \mathbf{N}}$ a sequence in M converging to x, such that $n^n d_g(x_n, x) \le \rho$ for some $\rho > 0$ and every $n \in \mathbf{N}$. There exists a smooth curve c in M such that $c(1/2^n) = x_n$ for every n and $c(0) = x$.*

Proof. The points x_n and x belong definitively, say for $n > \bar{n}$, to a normal chart $(U, \exp^{-1})$; we can assume that $x = \exp(0)$, $U = \exp B$ where B is an open ball in $T_x M$, so small that $d_g(\exp v, x) = \|v\|$, $v \in B$ (see for instance Theorem 5.7 Ch.VIII in [30]). Since the sequence $\{v_n\}_{n>\bar{n}}$, $v_n = \exp^{-1}(x_n)$, satisfies $n^n \|v_n\| \leq \rho$, we can construct a smooth curve γ in B with the properties that $\gamma(s) = 0$ for $s \leq 0$, $\gamma(1/2^n) = v_n$, and that $\gamma[1/2^{n+1}, 1/2^n]$ is the segment between v_{n+1} and v_n; by construction, γ is flat at every v_n (see Proposition 2.3.4 in [19]). The curve $s \rightsquigarrow c(s) = \exp \gamma(s)$ is well-defined and satisfies $c(1/2^n) = x_n$ for $n > \bar{n}$. As for the remaining points, first suppose that $\bar{n} = 1$ so that only the point x_1 does not belong to the curve c. Consider any curve c' in the interval $[1/4, +\infty)$ with the properties that $c'(s) = x_1$ for $s \geq 1/2$, $c'(1/4) = x_2$, with c' flat at x_2, and compose the curve c with c'. In the general case repeat the procedure adding all the remaining points. □

Lemma 9. *Let $M = \prod_{n \in \mathbf{N}} M_n$ be a product of connected manifolds and $\{x_k\}_{k \in \mathbf{N}}$ a sequence in M converging to x. Then there exist a subsequence $\{x_{k_r}\}$ and a C^∞ curve c in M such that $c(1/2^r) = x_{k_r}$ for every $r \in \mathbf{N}$, $c(0) = x$.*

Proof. Choose a metric g_n on every M_n and put $d_n(x_n, y_n) = d_{g_n}(x_n, y_n)(1 + d_{g_n}(x_n, y_n))^{-1}$, so that $d_n(x_n, y_n) \leq 1$ for $x_n, y_n \in M_n$. M is a metric space with the distance $d(x, y) := \sum_{n=1}^{+\infty} (1/2^n) d_n(x_n, y_n)$. Extract from $\{x_k\}$ a subsequence $\{x_{k_r}\}$ such that $\{r^r d(x_{k_r}, x)\}$ is bounded, so that even the sequence $\{r^r d_{g_n}(x_{k_r;n}, x_n)\}$ is bounded for each n. Using the Lemma 8 construct a smooth curve $c_n : \mathbf{R} \to M_n$, with $c_n(1/2^r) = x_{k_r,n}$ and $c_n(0) = x_n$, for every n. Then define $c(s) = \{c_n(s)\}_{n \in \mathbf{N}} \in M$. □

Lemma 10. *Let $M = \prod_{n \in \mathbf{N}} M_n$ be a product of connected manifolds. Then every $f \in C^\infty(M)$ is continuous.*

Proof. As M is metrizable we have only to prove that f is sequentially continuous. Assume that, for some sequence $\{x_k\}$ of M converging to x, there exists $\varepsilon > 0$ such that $|f(x_k) - f(x)| \geq \varepsilon$ for all k; by considering eventually a subsequence, construct by Lemma 9 a C^∞ curve c such that $c(1/2^k) = x_k$, $c(0) = x$. Then $f \circ c \in C^\infty(\mathbf{R}, \mathbf{R})$ and $f(x_k) = (f \circ c)(1/2^k) \to (f \circ c)(0) = f(x)$, contradicting the assumption. □

Let now $M = \prod_{t \in T} M_t$ and $q \in M$. For every subset $S \subset T$ we identify $M_S = \prod_{t \in S} M_t$ with $\{x \in M \mid x_t = q_t, \forall t \notin S\}$ and, for $x \in M$, we denote by x_S the element defined by $(x_S)_t = x_t$ if $t \in S$, $(x_S)_t = q_t$ if $t \in T - S$. Moreover we consider the subset M_0 of M consisting of the elements x with support $\{t \in T \mid x_t \neq q_t\}$ at most countable.

Lemma 11. *Let $M = \prod_{t \in T} M_t$ be a product of connected manifolds. Every $f \in C^\infty(M)$ is sequentially continuous on M_0.*

Proof. Let $x_k \to x$, with $x_k, x \in M_0$; there exists a subset $S \subset T$, S at most countable, containing the supports of x and of the x_k; the function $f \in C^\infty(M)$, if restricted to M_S, is a C^∞ function on M_S; then we apply Lemma 10 to get $f(x_k) \to f(x)$. □

The following theorem is a consequence of Mazur's results on product of metrizable spaces [34].

Theorem 12. *Let $M = \prod_{t\in T} M_t$ be a product of connected manifolds. Then every $f \in \mathcal{C}^\infty(M)$ is continuous and countably cylindrical.*

Proof. The restriction of f to M_0 is sequentially continuous. By Theorem II of [34] there exists a countable subset $S_f \subset T$ such that $f(x) = f(x_{S_f})$ for $x \in M_0$. We will prove that $f(x) = f(x_{S_f})$ for every x in M. We identify the space of the subsets of T with $\mathbf{2}^T$ endowed with the product topology and we prove that $\varphi_x : \mathbf{2}^T \to \mathbf{R}$, $\varphi_x(S) := f(x_S) - f(x_{S\cap S_f})$ is sequentially continuous. Let $S_n \to S$, so that, for large n, $(x_{S_n})_t = (x_S)_t$ holds for every $t \in T$. Applying Lemma 8, we can construct, for every $t \in T$, a smooth curve $c_t : \mathbf{R} \to M_t$ satisfying $c_t(1/2^n) = (x_{S_n})_t$ and $c_t(0) = (x_S)_t$. The curve $c = \{c_t\}_{t\in T}$ is a $\mathcal{C}^\infty$ curve and satisfies $c(1/2^n) = x_{S_n}$, $c(0) = x_S$. Since $f \circ c \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$, we get $f(x_{S_n}) \to f(x_S)$, proving that φ_x is sequentially continuous. By Theorem III of [34] we conclude that φ_x is continuous. For every finite set S, we have $\varphi_x(S) = 0$ and, applying the results in Section 1, Example 3 of [34], we obtain that $\varphi_x(S) = 0$, for every subset S of T.

Then $f = \pi^*_{S_f} f_{S_f}$, where f_{S_f} is the restriction of f to M_{S_f}. Continuity of f follows by Lemma 10. □

Now we come to the problem of defining the differential of $\mathcal{C}^\infty$ functions on products of manifolds. In the case of $\mathbf{R}^T$ every $f \in \mathcal{C}^\infty(\mathbf{R}^T)$ admits a differential: as f is countably cylindrical, we are reduced to the case of $f \in \mathcal{C}^\infty(\mathbf{R}^\mathbf{N})$ discussed in Proposition 7, obtaining $\mathcal{C}^\infty(\mathbf{R}^T) = \mathcal{C}yl^\infty_\ell(\mathbf{R}^T)$. Therefore the differential of f is $\mathbf{d}f$, as defined in Section 2, it is continuous and satisfies the chain rule. Now we shall prove that a similar property holds for $\mathcal{C}^\infty$ functions on products of manifolds $M = \prod_{t\in T} M_t$. For a curve c in M we put

$$\dot{c}(s) := \{\dot{c}_t(s)\}_{t\in T} \in TM.$$

Theorem 13. *Let $M = \prod_{t\in T} M_t$ be a product of connected geodesically complete Riemannian manifolds. Then every $f \in \mathcal{C}^\infty(M)$ is locally cylindrical.*

Proof. In an ordinary complete Riemannian manifold we denote by $\gamma_{x,v}$ the geodesic curve starting from the point x with velocity v and by Φ the flow of the spray defined by the metric. We recall that $\gamma_{x,v}(s) = \tau(\Phi(s, v_x))$, where τ is the tangent projection and $v_x \equiv (x, v)$, and that $\dot{\gamma}_{x,v}(s) = \Phi(s, v_x)$. By $\Phi(s+h, v_x) = \Phi(h, \Phi(s, v_x))$, we have

$$\gamma_{x,v}(s+h) = \gamma_{\gamma_{x,v}(s),\dot{\gamma}_{x,v}(s)}(h) \tag{2}$$

for every $s, h \in \mathbf{R}$.

We come now to $M = \prod_{t\in T} M_t$. For $x \in M$, $v \in T_xM$ and $s \in \mathbf{R}$, we denote by $\gamma_{x,v}(s)$ the product $\{\gamma_{x_t,v_t}(s)\}_{t\in T}$, where γ_{x_t,v_t} are geodesic curves in M_t, and call *geodesic curve* at x with velocity v the curve $\gamma_{x,v} : \mathbf{R} \to M$, $s \rightsquigarrow \gamma_{x,v}(s)$. The geodesic curve $\gamma_{x,v}$ satisfies formula (2).

We define now $df : TM \to \mathbf{R}$ by

$$df\,(x, v) := (f \circ \gamma_{x,v})^{\cdot}\,(0)\,. \tag{3}$$

Let $s \rightsquigarrow (x(s), v(s))$ be a curve in TM. Applying Boman Theorem one easily recognizes that the map $\varphi : \mathbf{R}^2 \to \mathbf{R}$, $\varphi(s, h) := f(\gamma_{x(s),v(s)}(h))$ is smooth. Therefore the map $s \rightsquigarrow \frac{\partial}{\partial h}\varphi_{(s,0)} = df(x(s), v(s))$ is smooth. This proves that df is a C^∞ map, so that df is continuous by Theorem 12.

Using the Hopf–Rinow Theorem to each component, we get that for every $x, y \in M$ there exists a geodesic curve γ (possibly not unique) joining x to y, so that

$$f\,(y) - f\,(x) = \int_0^1 (f \circ \gamma)^{\cdot}\,(s)\,\mathrm{d}s.$$

We prove that $(f \circ \gamma)^{\cdot}\,(s) = df\,(\gamma\,(s)\,, \dot{\gamma}\,(s))$. One has indeed by formulae (2) and (3)

$$\lim_{h\to 0} \frac{1}{h}\,(f\,(\gamma\,(s+h)) - f\,(\gamma(s)))$$
$$= \lim_{h\to 0} \frac{1}{h}\left(f\left(\gamma_{\gamma(s),\dot{\gamma}(s)}\,(h)\right) - f\,(\gamma\,(s))\right) = df\,(\gamma\,(s)\,, \dot{\gamma}\,(s))\,.$$

Then we remark that $df(x, rv) = r df(x, v)$ for every $v \in T_xM$ and $r \in \mathbf{R}$ (one can use simply a reparametrization of curves) and, in particular, that $df(x, 0) = 0$. Therefore the subset W of TM on which $|df(x, v)| < 1$ is an open neighborhood of the zero section.

We fix $x_0 \in M$ and construct an open set $U \subset W$ of the form $U = \prod_{t\in T} U_t$, with $U_t = TM_t$ except for a finite set T_0 of indices, and such that $x_0 \in \tau(U)$. If $(x, v) \in U$ with $v_t = 0$ for $t \in T_0$, then also $(x, rv) \in U$ for every $r \in \mathbf{R}$, and the condition $|df(x, rv)| = |r df(x, v)| < 1$ for every $r \in \mathbf{R}$ implies that $df(x, v) = 0$. The set $V := \tau(U)$ is an open neighbourhood of x_0. If $x, y \in V$ satisfy $x_t = y_t$ for $t \in T_0$, there exists a geodesic curve γ in V joining x to y whose components γ_t are constant for $t \in T_0$. Then

$$f(y) - f(x) = \int_0^1 df(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s = 0.$$

This proves that f is cylindrical on V, i.e. $f_{\restriction V} = \pi^*_{T_0} g_{\restriction V}$ with $g \in C^\infty\left(\pi_{T_0} V\right)$. □

Every ordinary manifold admits a complete metric [42], so we get the following result.

Theorem 14. *Let $M = \prod_{t\in T} M_t$ be a product of connected manifolds. A function f on M belongs to $C^\infty\,(M)$ if and only if it belongs to $Cyl^\infty_\ell(M)$. If the factors M_t are also compact, then every $f \in C^\infty(M)$ is cylindrical.*

Remark. The above theorem is a version of Boman Theorem characterizing locally cylindrical smooth functions on products M of connected manifolds and proves that the C^∞

calculus introduced by Fröhlicher and Kriegl agrees with the differential calculus proposed by Ashtekar and Lewandowski in the case of products of compact connected manifolds. In particular, the kinematical tangent space $\mathcal{T}M$ agrees with the tangent space TM and, for each $f \in \mathcal{C}^{\infty}(M) \equiv Cyl_{\ell}^{\infty}(M)$ and $x \in M$, the kinematical differential $\delta_x f$ agrees with the differential $\mathbf{d}_x f$ defined in Section 2, so that

$$\delta_{c(s)} f(c(s), \dot{c}(s)) = (f \dot{\circ} c)(s) \quad \forall s \in \mathbf{R}$$

for every curve c. Moreover, the $\mathcal{C}^{\infty}$ functions are continuous and admit iterated differentials.

When some factor M_t is not connected, the product M is not connected. However, Theorem 13 applies to each connected component (of M), which results a product of connected manifolds. In this setting the $\mathcal{C}^{\infty}$ functions on M could not be continuous, but they are locally cylindrical (hence continuous) on each component.

Example 9. An interesting example of product of compact Lie groups has been proposed as a compact extension of the group $\mathcal{G}$ of gauge transformations of a principal bundle $P(B, G)$ with compact connected gauge group G in [5]. We recall that $\mathcal{G}$ is the group of smooth sections of the associated bundle $P[G]$ on B, whose fibre on $x \in B$ is a group G_x isomorphic with G. The group $\mathcal{G}$ is naturally included in $\overline{\mathcal{G}} = \prod_{x \in B} G_x$ by $\eta : \mathcal{G} \to \overline{\mathcal{G}}$, $\eta(g) := \{g(x)\}_{x \in B}$. Assuming the cardinality of B (space or space–time) to be $\leq 2^{\aleph_0}$ (i.e. assuming the continuum hypothesis) we obtain that every $\mathcal{C}^{\infty}$-function on $\overline{\mathcal{G}}$ is continuous and cylindrical (Theorem 13).

The group $\mathcal{G}$, a natural structure of infinite-dimensional Lie group can be given. If B is compact, $\mathcal{G}$ is a Lie group modelled on a nuclear Fréchet space. As remarked in Section 3, this implies that a $\mathcal{C}^{\infty}$ structure for $\mathcal{G}$ is given, admitting $C_c^{\infty}(\mathbf{R}, \mathcal{G})$ as structure curves and $C_c^{\infty}(\mathcal{G}, \mathbf{R})$ as structure functions.

Proposition 15. *Let $P(B, G)$ a principal bundle with B and G compact. The inclusion $\eta : \mathcal{G} \hookrightarrow \overline{\mathcal{G}}$ is a $\mathcal{C}^{\infty}$ continuous (but not open) map. Its image is dense.*

Proof. First we prove that η is $\mathcal{C}^{\infty}$, i.e. that images of curves in $\mathcal{G}$ are curves in $\overline{\mathcal{G}}$. Let $s \rightsquigarrow g(s)$ be a curve in $\mathcal{G}$. We have to prove that for every $x \in B$ the curve $s \rightsquigarrow (g(s))(x) \in G_x$ is smooth. This is true since the projection π_x, if restricted to $\mathcal{G}$, agrees with the evaluation map $ev_x : \mathcal{G} \to G_x$ which is C_c^{∞} by Corollary 11.7 of [35]. This also implies that the inclusion is continuous.

To prove density we only observe that, given a finite set $\{x_i\} \subset B$ and $g_i \in G_{x_i}$, there exists $g \in \mathcal{G}$ with $g_{x_i} = g_i$ for every i.

Completeness of $\mathcal{G}$ implies that every homeomorphic image of $\mathcal{G}$ in a topological group is closed, hence the inclusion $\mathcal{G} \hookrightarrow \overline{\mathcal{G}}$ cannot be open. □

We stress, however, that the group $\overline{\mathcal{G}}$ is not a compactification of $\mathcal{G}$ endowed with the topology of Fréchet Lie group.

5. Projective limits of manifolds

The category of $\mathcal{C}^\infty$ spaces is closed with respect to projective limits. In particular, the limit M of a projective family of manifolds $\{M_j, \pi_{ij}, J\}$ admits a canonical $\mathcal{C}^\infty$ structure, where the set of structure curves is

$$\mathcal{C} := \{c : \mathbf{R} \to M;\ \pi_j \circ c \in C^\infty(\mathbf{R}, M_j)\ \forall j \in J\}.$$

This is precisely the set of $\mathcal{C}^\infty$ curves in $\prod_{j\in J} M_j$ laying in $M \subset \prod_{j\in J} M_j$, so that

$$Cyl^\infty(M) \subset Cyl^\infty_M\left(\prod_{j\in J} M_j\right) \subset \mathcal{C}^\infty_M\left(\prod_{j\in J} M_j\right) \subset \mathcal{C}^\infty(M),$$

where $\mathcal{C}^\infty_M(\prod_{j\in J} M_j)$ denotes the ring of the restrictions to M of $\mathcal{C}^\infty$ functions on $\prod_{j\in J} M_j$; the ring $Cyl^\infty_M(\prod_{j\in J} M_j)$ is analogously defined.

Proposition 16. *Let $M = \varprojlim_{j\in J} M_j$ be a projective limit of manifolds. Then*
(1) *$Cyl^\infty(M) = Cyl^\infty_M(\prod_{j\in J} M_j)$;*
(2) *if the factors M_j are compact connected manifolds, a $\mathcal{C}^\infty$ function on M is cylindrical if and only if it admits a $\mathcal{C}^\infty$ extension to $\prod_{j\in J} M_j$.*

Proof. (1) Let $f \in Cyl^\infty(M)$, $f = \pi_j^* f_j$ for some $f_j \in C^\infty(M_j)$. Define $f^\sharp = p_j^* f_j$ where $p_j : \prod_{\iota\in J} M_\iota \to M_j$ is the Cartesian projection. One easily checks that $f^\sharp$ is well-defined and that $f = f^\sharp \circ i_M$, where $i_M : M \to \prod_{j\in J} M_j$ is the canonical inclusion. Consider now any smooth cylindrical function h on $\prod_{j\in J} M_j$, with $h = p_{J_0}^* h_0$, $J_0 = \{j_1, \ldots, j_k\}$ and $h_0 : \prod_{j\in J_0} M_j \to \mathbf{R}$ smooth. It is easy to prove that $f = i_M^* h$ is cylindrical. Actually, choose $\tilde{j}$ dominating J_0 and define $f_{\tilde{j}} : M_{\tilde{j}} \to \mathbf{R}$, $f_{\tilde{j}}(\tilde{x}) = h_0(\pi_{\tilde{j},j_1}(\tilde{x}), \ldots, \pi_{\tilde{j},j_k}(\tilde{x}))$. Then check that $f = \pi_{\tilde{j}}^* f_{\tilde{j}}$.

(2) This is an immediate consequence of (1) and of Theorem 13. □

The ring $Cyl^\infty(M)$ is a generating set of functions for the canonical $\mathcal{C}^\infty$ structure and appears just a minimal choice for the ring of smooth functions.

The consistency of $\mathcal{C}^\infty(M)$ for a projective limit of manifolds M could be a problem not so easily estabilished as in the case of products of manifolds discussed in the above section. The main reason is that the paucity of $\mathcal{C}^\infty$ curves produces a plenty of $\mathcal{C}^\infty$ functions. Even if projective limits of compact connected manifolds are connected, they could not be path connected, not even locally path connected (see for instance Σ_p, Σ_∞, E_∞ and H_∞ discussed in Sections 2 and 3). If M admits many path components, there exist $\mathcal{C}^\infty$ functions on M which are not continuous, hence not cylindrical.

If M is the limit of compact connected manifolds M_j, then every $\mathcal{C}^\infty$ function admitting a $\mathcal{C}^\infty$ extension to $\prod_{j\in J} M_j$ is cylindrical, hence continuous. One can ask whether each continuous $\mathcal{C}^\infty$ function f on M is cylindrical. Tiesze Extension Theorem assures that f admits a continuous extension $\tilde{f}$ to $\prod_{j\in J} M_j$. This extension could not be a $\mathcal{C}^\infty$ map, hence one cannot assure that f is cylindrical. An example is given on Σ_p (see later). Anyway, Theorem II of [34] assures that $\tilde{f}$, hence f, is countably cylindrical.

Now we briefly discuss tangent space. Obviously, $TM = \varprojlim_{j\in J} TM_j$ is a $\mathcal{C}^\infty$ space and the projection $\tau : TM \to M$ is a $\mathcal{C}^\infty$ map.

As M is a $\mathcal{C}^\infty$ space, it admits also a kinematical tangent space $\mathcal{T}M$. A good functoriality would require that $\mathcal{T}M$ agrees with TM, as in the case of products. This condition allows us to differentiate every $\mathcal{C}^\infty$ function on TM. If $\mathcal{C}^\infty(M) = Cyl^\infty_\ell(M)$, the tangent spaces agree, but this condition is not necessary, as we shall see discussing the examples below.

$J^\infty(M, N)$. A simple example of projective limit of a surjective family of non compact manifolds is the space $J^\infty(M, N)$ introduced in Section 2, Example 2, which is a Fréchet manifold modelled on a nuclear Fréchet space, the product of a sequence of finite dimensional vector spaces. By Theorem A.2 in Appendix A, the $\mathcal{C}^\infty$ functions on $J^\infty(M, N)$ are precisely the C^∞_c functions. Each local expression of a C^∞_c function f on $J^\infty(M, N)$ is locally cylindrical by Theorem 14, so that f itself is locally cylindrical.

Of course, even in this case the restriction to smooth cylindrical functions appears to be unnecessary.

The universal laminations. We return to the spaces Σ_p, Σ_∞, E_∞ and H_∞ introduced in Section 2. These spaces are foliated spaces and projective limits of manifolds. Accordingly, they admits two canonical $\mathcal{C}^\infty$ structures. Luckily, these $\mathcal{C}^\infty$ structures agree. Let M stand for Σ_p, Σ_∞, E_∞ or H_∞ and $\{M_j, \pi_j, J\}$ for the corresponding projective family of manifolds. We have to prove that C^∞_l curves in M are precisely the paths $c : \mathbf{R} \to M$ such that all $\pi_j \circ c$ are smooth. Let c be a C^∞_l curve in M. Then the projection of c in M_{j_0} is smooth, where j_0 denotes the minimum of J, as one can easily prove using the foliated atlas given in Section 3. Since each π_{j,j_0} is a covering of M_{j_0}, even the projection of c on M_j is smooth. Conversely, let $c = \{c_j\}_{j\in J}$ be a thread of smooth curves, then c is continuous and contained in a leaf, since leaves are the path components. Composing c with the foliated charts we get that c is a C^∞_l curve.

Coming to $\mathcal{C}^\infty$ functions, we immediately see that

$$Cyl^\infty(M) \subset C^\infty_l(M) \subset \mathcal{C}^\infty(M).$$

We recall that the last inclusion is proper (see Example 7, Section 3). To show that even the first inclusion can be proper, define $f : \mathbf{R} \times \Delta_p \to \mathbf{R}$ by

$$f(t, \mathbf{x}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sin\left(\frac{2\pi}{p^n}(t - (x_0 + x_1 p + \cdots + x_{n-1} p^{n-1}))\right).$$

The function f is a uniform limit of linear combinations of characters, so it is continuous. One easily recognizes that its quotient map $\tilde{f} : \Sigma_p \to \mathbf{R}$ is well-defined and a C^∞_l map. However, $\tilde{f}$ is not a cylindrical map.

Coming to tangent spaces we see that

$$\mathcal{T}M = T_l M = TM$$

as $\mathcal{C}^\infty$ spaces. The first equality was proved in Section 3. We have to prove that $TM = T_l M$. For every $x \in M$ we have $T_x M = \varprojlim_{j\in J} T_{x_j} M_j \simeq T_x \mathcal{L}_x$ since $\varprojlim_{j\in J} T_j M_j$ is a trivial limit and $\mathcal{L}_x$ is a covering of every M_j. To show that $TM \simeq T_l M$ as $\mathcal{C}^\infty$ spaces we can use the same arguments we used above to prove that $\mathcal{C}^\infty$ curves and C_l^∞ curves on M agree.

In this example the lack of path connectedness yields a huge quantity of $\mathcal{C}^\infty$ functions. Nevertheless, this excess of $\mathcal{C}^\infty$ functions does not create serious problems for differential calculus. Actually, each $\mathcal{C}^\infty$ function is differentiable on TM owing to the fact that the various notions of tangent space agree. We see therefore that $\mathcal{C}^\infty$ differential calculus can work even if the lack of continuity for $\mathcal{C}^\infty$ functions could be an unpleasant aspect. In this example the relevant ring of functions appears to be $C_l^\infty(M)$, which lies between $Cyl^\infty(M)$ and $\mathcal{C}^\infty(M)$.

Projective limits of manifolds in gauge theories. In the loop quantization of 2D Yang–Mills Theories and Loop Quantum Gravity the tool of projective limit has been proven useful to embed the configuration space $\mathcal{A}/\mathcal{G}$ of the theory in a compact space $\overline{\mathcal{A}/\mathcal{G}}$ on which measures are defined suitable for quantization. Here $\mathcal{A}$ denotes the space of principal connections of a principal bundle $P(B, G)$, with G a compact connected group and $\mathcal{G}$ denotes the group of gauge transformations. In the literature many proposals of this procedure can be found, whose starting point is a suitable family of multiloops, graphs or lattices, used as index set for a projective family. Here we briefly discuss the projective limits of manifolds introduced in [5].

Let B be a real analytic connected manifold. By a *parametrized edge* we mean a homeomorphism e from $[0, 1]$ into B such that $e_{\restriction(0,1)} \to B$ is an analytic embedding. An *unparametrized edge* is an equivalence class of parametrized edges with respect to reparametrization by analytic bijections of the interval $[0, 1]$. The end points of an edge e, called the *vertices* of e, and the range e^* do not depend by such reparametrizations. A *graph* γ in M consists of finitely many unparametrized edges e_i, such that

(1) for $e_i \neq e_j$, $e_i^* \cap e_j^*$ is contained in the set of vertices of e_i and e_j;

(2) every edge of γ is at both sides connected with another edge.

The set L of all the graphs in M can be given a partial order, where $\gamma_1 \leq \gamma_2$ whenever each edge of γ_1 can be expressed as a composition of edges of γ_2 and each vertex in γ_1 is a vertex of γ_2. Due to analyticity of edges, L is a directed set.

For every edge e, denote by $\widehat{\mathcal{G}}_e$ the closed normal subgroup of $\mathcal{G}$ consisting of gauge transformations acting as the identity over the vertices of e. Define an equivalence relation $\sim_e$ on $\mathcal{A}$ by

$$A \sim_e A' \quad \text{if } A_{\restriction e^*} = A'_{\restriction e^*} \bmod \widehat{\mathcal{G}}_e.$$

We denote by $\mathcal{A}_e$ the quotient space and by $\pi_e : \mathcal{A} \to \mathcal{A}_e$ the canonical projection. It is well known that, for a given orientation on e, the parallel transport along e defined by a connection A, denoted $\mathcal{P}_e{}^A$, belongs to $Eq(P_{e(0)}, P_{e(1)})$, the space of G equivariant maps from the fibre $P_{e(0)}$ to the fibre $P_{e(1)}$. The parallel transport map $\mathcal{P}_e : \mathcal{A} \to Eq(P_{e(0)}, P_{e(1)})$ quotients to a one-to-one map $\Lambda_e : \mathcal{A}_e \to Eq(P_{e(0)}, P_{e(1)})$. By means of Λ_e, a (analytic) manifold structure on $\mathcal{A}_e$ can be given, which does not depend on the chosen orientation: for

$x, x' \in B$ the space $Eq(P_x, P_{x'})$ is a compact manifold diffeomorphic to G and is canonically diffeomorphic to $Eq(P_{x'}, P_x)$. For a graph γ, one considers the compact connected manifold

$$\mathcal{A}_\gamma := \prod_{e\in\gamma} \mathcal{A}_e$$

and the projection $\pi_\gamma : \mathcal{A} \to \mathcal{A}_\gamma$, $\pi_\gamma := \prod_{e\in\gamma} \pi_e$. For $\gamma < \gamma'$ a projection $\pi_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$ is defined by $\pi_{\gamma'\gamma} \circ \pi_{\gamma'} = \pi_\gamma$. This gives a projective surjective family of compact connected manifolds whose limit $\overline{\mathcal{A}}$ is a compact connected space containing $\mathcal{A}$ as dense subset. Elements of $\overline{\mathcal{A}}$ are called generalized connections. Analogous constructions can be given using suitably defined oriented edges and oriented graphs.

The affine space $\mathcal{A}$ of the smooth connections is modelled on a nuclear Fréchet space, in the case where B is compact (for the case where B is not compact, see [1]). The inclusion of $\mathcal{A}$ in $\overline{\mathcal{A}}$ is $\mathcal{C}^\infty$ and continuous, but it is neither a homeomorphism nor a $\mathcal{C}^\infty$ diffeomorphism with its image. This holds also for the inclusions of the various Sobolev completions of $\mathcal{A}$ used in the literature. In this sense $\overline{\mathcal{A}}$ is not a true compactification.

A projective family of Lie groups $\{\mathcal{G}_\gamma, \pi_\gamma, L\}$ is also introduced where $\mathcal{G}_\gamma := \mathcal{G}/\widehat{\mathcal{G}}_\gamma$, with $\widehat{\mathcal{G}}_\gamma := \bigcap_{e\in\gamma} \widehat{\mathcal{G}}_e$ and π_γ is the canonical projection. The projective limit of this projective family of Lie groups is precisely the group $\overline{\mathcal{G}}$ considered in Section 3. The action of $\mathcal{G}$ on $\mathcal{A}$ extends to an action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$.

In gauge theories the primary object would be $\overline{\mathcal{A}/\mathcal{G}}$, the limit of the projective family of orbit spaces $\mathcal{A}_\gamma/\mathcal{G}_\gamma$, also considered in [5]. These orbit spaces fail to be genuine manifolds in general. However, the authors proved that $\overline{\mathcal{A}/\mathcal{G}}$ is homeomorphic to $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ so that a differential calculus can be defined on $\overline{\mathcal{A}/\mathcal{G}}$ by means of $\overline{\mathcal{G}}$-equivariant cylindrical smooth maps on $\overline{\mathcal{A}}$.

The comparison of $\mathcal{C}^\infty$ functions with cylindrical smooth functions on $\overline{\mathcal{A}}$ is a delicate problem, due to the complexity of the index set and nontriviality of projection maps. Even the investigation of the path connectedness of $\overline{\mathcal{A}}$ could reveal a nontrivial problem. For the Abelian case a general method is reported in Appendix B.

One could hope that the projective limit $\overline{\mathcal{A}}$ shares some features with the universal laminations. Even in this case indeed the projective family is obtained taking quotients of the same flat space. However, the treatment of these limits requires techniques beyond the ones developed in this paper. Moreover, the space $\overline{\mathcal{A}}$ could be too large for the needs of Quantum Field Theory. Actually, some projective subfamilies (as lattices) or other projective families (based on multiloops or spin networks instead of graphs) are used in the literature, to get analogous compactifications of $\overline{\mathcal{A}}$. Physical and mathematical criteria have to be adopted to select a convenient compactification. A good mathematical requirement could be to dispose of a suitable Boman Theorem to get a fine differential calculus.

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Appendix A

It is well known that standard differential calculus works well for finite dimensional vector spaces and for Banach spaces and that a lot of inequivalent differential calculi can be given in general locally convex vector spaces. However, nearly all the main notions of infinite differentiability agree in Fréchet spaces [7,24] with the C_c^∞ differentiability defined as follows.

Let $U \subset E$ be an open subset of a complete locally convex vector space. A mapping $f : U \to F$ is said to be C_c^1 on U if the following conditions hold:

(1) $\lim_{h\to 0}(1/h)(f(x+hy) - f(x)) = Df(x)y$ where $Df(x) : E \to F$ is a linear map, for $x \in U$, $y \in E$, $h \in \mathbf{R}$.

(2) The map $Df : U \times E \to F$, $(x, y) \rightsquigarrow Df(x)y$ is jointly continuous.

The set of these mappings is denoted by $C_c^1(U, F)$. The spaces $C_c^k(U, F)$, $k > 1$, are defined by recursion, as the set of the maps in $C_c^{k-1}(U, F)$ such that $D^{k-1}f : U \times E^{k-1} \to F$ is C_c^1. Then $C_c^\infty(U, F) := \bigcap_{k\geq 1} C_c^k(U, F)$.

More results on C_c^∞ calculus can be found in [24] or [36]. In Fréchet spaces the C_c^∞ calculus agrees even with the $\mathcal{C}^\infty$ calculus. We give the proof of this statement which one can find in [19], entangled with more general results. A similar procedure has been adopted to obtain the results in Section 4. We recall that in a Fréchet space E the structure curves are precisely the C_c^∞ curves.

Lemma A.1. *Let E be a Fréchet space and f a $\mathcal{C}^\infty$ function on E. Then f is continuous.*

Proof. Suppose f is not continuous at x. Then there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ converging to x such that $|f(x_n) - f(x)| \geq \epsilon$ for some $\epsilon > 0$. Extract from $\{x_n\}_{n\in\mathbf{N}}$ a subsequence $\{x_{n_k}\}$ such that $\{k^k d(x, x_{n_k})\}$ is bounded, where d is a distance on E generating the topology of E. Appying Lemma 2.3.4 of [19], construct a curve c in E such that $c(0) = x$ and $c(1/2^k) = x_{n_k}$ for every k. The assumption $f \in \mathcal{C}^\infty(E, \mathbf{R})$ would imply $f(x_{n_k}) \to f(x)$, giving a contradiction. □

The following theorem is the Boman Theorem for C_c^∞ calculus on Fréchet spaces.

Theorem A.2. *Let E be a Fréchet space and $f : E \to \mathbf{R}$. The following statements are equivalent:*

(1) f is a $\mathcal{C}^\infty$ function;

(2) f is a C_c^∞ function.

Proof. (1) implies (2): For $x, y \in E$ the map $h \rightsquigarrow f(x + hy)$ belongs to $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$. We shall prove that the map $df : E \times E \to \mathbf{R}$ defined by

$$df(x, y) = \lim_{h\to 0} 1/h(f(x + hy) - f(x))$$

is a $\mathcal{C}^\infty$ map, jointly continuous and linear in the second variable.

To get that df is a $\mathcal{C}^\infty$ map, we have to prove that the map $\varphi : t \rightsquigarrow \varphi(t) := df(x(t), y(t))$ is smooth, for every pair of curves $t \rightsquigarrow x(t)$, $t \rightsquigarrow y(t)$ on E. Consider the $\mathcal{C}^\infty$ map $\Phi : \mathbf{R}^2 \to \mathbf{R}$ defined by $\Phi(t, h) = f(x(t) + hy(t))$. The Boman Theorem on $\mathbf{R}^2$ gives $\Phi \in C^\infty(\mathbf{R}^2, \mathbf{R})$. Since

$$\frac{\partial}{\partial h}\Phi(t, h)|_{h=0} = df(x(t), y(t)) = \varphi(t),$$

the map φ is smooth. Therefore df is a $\mathcal{C}^\infty$ map.

By Lemma A.1 df is continuous. Obviously, df is homogeneous in the second variable. Hence it is linear by Proposition 4.4.22 of [19].

We have proved that $f \in C^1_c(E, \mathbf{R})$ with $df \in \mathcal{C}^\infty(E \times E, \mathbf{R})$. By recursivity, this proves that $f \in C^\infty_c(E, \mathbf{R})$.

(2) implies (1): For $f \in C^\infty_c(E, \mathbf{R})$ and every curve $c \in C^\infty_c(\mathbf{R}, E)$, the composition $f \circ c \in C^\infty_c(\mathbf{R}, \mathbf{R}) = C^\infty(\mathbf{R}, \mathbf{R})$, so that f is a $\mathcal{C}^\infty$ function. □

Appendix B

Here we refer to the last example in Section 5 and investigate the path connectedness of $\overline{\mathcal{A}}$ for $G = U(1)$. As proved in [4], one can reduce to a trivial principal bundle $P = B \times U(1)$ so that $\mathcal{A} = A^1(B)$, the space of smooth 1-forms on B. The group $\mathcal{G} = C^\infty(B, U(1))$ acts on $\mathcal{A}$ by translations $A_\mu \rightsquigarrow A_\mu + g^{-1}\partial_\mu g$, so that the action defines a homomorphism of the Abelian group $\mathcal{G}$ in the Abelian group $\mathcal{A}$. Thus also $\mathcal{A}/\mathcal{G}$ is an Abelian group.

The triviality of P and commutativity of $U(1)$ imply that $\overline{\mathcal{G}} = U(1)^B$, that $\mathcal{A}_\gamma$ is canonically isomorphic to $U(1)^{E(\gamma)}$ (where $E(\gamma)$ is the number of edges of γ) and that the projections $\pi_{\gamma,\gamma'}$ are group homomorphisms. Hence, $\overline{\mathcal{A}}$ is a compact connected Abelian group. Moreover, there exists a short exact sequence of compact connected Abelian groups

$$0 \to U(1) \to \overline{\mathcal{G}} \to \overline{\mathcal{A}} \to \overline{\mathcal{A}}/\overline{\mathcal{G}} \to 0\,. \tag{B.1}$$

We summarize some of the classical results given in [16] about path connectedness of compact connected Abelian groups.

Proposition B.1. *Let X be a compact connected Abelian group. Then the dual group $X^\dagger$ is discrete and torsion free. The following conditions are equivalent:*
(1) *X is path connected;*
(2) $Ext^1_{\mathbf{Z}}(X^\dagger, \mathbf{Z}) = 0$;
(3) *every element of X is of the form $e^{i\lambda}$ where $\lambda \in \mathrm{Hom}(X^\dagger, \mathbf{R})$.*

If $X^\dagger$ is countable, the above conditions are equivalent to the requirement that $X^\dagger$ is free.

For every graph γ, the dual group $A_\gamma^\dagger$ of $\mathcal{A}_\gamma$ is the free group generated by the edges in γ, provided that to every edge e_k of γ the character

$$\chi_{e_k} : \mathcal{A}_\gamma \to U(1), \quad \chi_{e_k}(A_\gamma) := \mathrm{e}^{\mathrm{i}\int_{e_k} A}$$

is associated, where $A \in \mathcal{A}$ is any representative of A_γ. The dual group $\overline{\mathcal{A}}^\dagger$ of $\overline{\mathcal{A}}$ is the direct limit of the dual groups $\mathcal{A}_\gamma^\dagger$. Every character χ of $\overline{\mathcal{A}}$ belongs to some $\mathcal{A}_\gamma^\dagger$, so that $\chi = \sum_{e_k \in \gamma} n_k \chi_{e_k}$ and for $\bar{A} \in \overline{\mathcal{A}}$ we have

$$\langle \bar{A}, \chi \rangle = \langle A_\gamma, \chi \rangle = \prod_k {\chi_{e_k}}^{n_k}(A_\gamma).$$

In particular, if $\bar{A} = A$ is a smooth connection, it verifies

$$\langle A, \chi \rangle = \mathrm{e}^{\mathrm{i}\lambda(\chi)},$$

where $\lambda \in \mathrm{Hom}(\overline{\mathcal{A}}^\dagger, \mathbf{R})$ is defined by $\lambda(\chi) = \sum_{e_k \in \gamma} n_k \int_{e_k} A$. Also the examples of generalized connections given in [4] are of the form $\mathrm{e}^{\mathrm{i}\lambda}$ with $\lambda \in \mathrm{Hom}(\overline{\mathcal{A}}^\dagger, \mathbf{R})$, so one can hope that condition (3) is always verified.

Utilizing the exact sequence

$$0 \to \overline{\overline{\mathcal{G}}} \to \overline{\mathcal{A}} \to \overline{\mathcal{A}/\mathcal{G}} \to 0\,,$$

where $\overline{\overline{\mathcal{G}}} := \overline{\mathcal{G}}/U(1)$, and Proposition 4, Section 5.5 in [29], we obtain the exact sequence

$$Ext^1_{\mathbf{Z}}((\overline{\mathcal{A}/\mathcal{G}})^\dagger, \mathbf{Z}) \to Ext^1_{\mathbf{Z}}(\overline{\mathcal{A}}^\dagger, \mathbf{Z}) \to Ext^1_{\mathbf{Z}}(\overline{\overline{\mathcal{G}}}^\dagger, \mathbf{Z}) \to 0,$$

where $Ext^1_{\mathbf{Z}}(\overline{\overline{\mathcal{G}}}^\dagger, \mathbf{Z}) = 0$, since the Abelian group $\overline{\overline{\mathcal{G}}}$ is compact and path connected. Therefore $Ext^1_{\mathbf{Z}}((\overline{\mathcal{A}/\mathcal{G}})^\dagger, \mathbf{Z}) = 0$ would imply that $Ext^1_{\mathbf{Z}}(\overline{\mathcal{A}}^\dagger, \mathbf{Z}) = 0$. This proves that $\overline{\mathcal{A}}$ is path connected if and only if $\overline{\mathcal{A}/\mathcal{G}}$ is path connected.

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